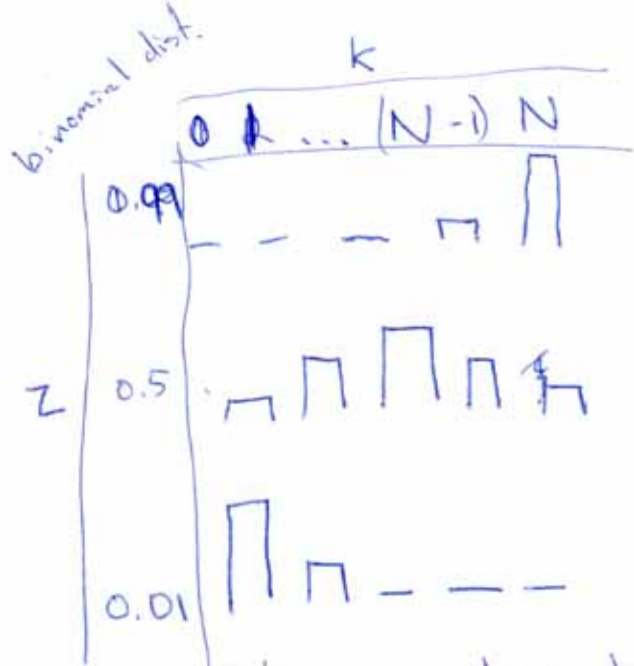
Problem: Estimate true germination rate, z, when planting a finite number of seeds, n, of which k seeds germinate. What is the credible interval boundary for n, k?

binomial

$$\text{PMF} = \binom{n}{k} p^k (1-p)^{n-k}$$



normalize over $k = \text{const}$

normalization for $k = \text{const}$

$$A = \binom{n}{k} \int_0^1 p^k (1-p)^{n-k} \, dp$$

$$\text{let } B = \int_0^1 p^k (1-p)^{n-k} \, dp$$

$$\boxed{P = \frac{1}{B} z^k (1-z)^{n-k}}$$

normalized, constant-k PMF for z, given n, k

$$B = \frac{\Gamma(k+1)\,\Gamma(-k+n+1)}{\Gamma(n+2)} = \cancel{\frac{(k+1)!\,(1+n-k)!}{(n+2)!}}$$

$$\frac{1}{B} = \frac{\Gamma(n+2)}{\Gamma(k+1)\,\Gamma(-k+n+1)} \rightarrow \boxed{\text{PMF} = \frac{z^k (1-z)^{n-k}\,\Gamma(n+2)}{\Gamma(k+1)\,\Gamma(-k+n+1)}}$$

$$\text{mean} = \int_0^1 z \, \text{PMF}(z) \, dz$$

continued

Mean

$$\mu = \int_0^1 \left[z^{k+1} (1-z)^{n-k} \right] \left(\frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(-k+n+1)} \right)$$

$$\mu = \frac{\Gamma(k+2)\cancel{\Gamma(-k+n+1)}}{\Gamma(n+3)} \left[\frac{\Gamma(n+2)}{\Gamma(k+1)\cancel{\Gamma(-k+n+1)}} \right]$$

$$\cancel{\mu = \frac{\Gamma(k+2)}{\Gamma(n+3)}} \quad ?$$

$$\mu = \frac{\Gamma(k+2)\,\Gamma(n+2)}{\Gamma(n+3)\,\Gamma(k+1)}$$

$$\mu = \frac{(k+1)!\,(n+1)!}{(n+2)!\,(k)!} = \frac{k+1}{n+2}$$

$$\boxed{\mu = \frac{k+1}{n+2}}$$

$$\text{CDF} = \int_0^z \frac{x^k (1-x)^{n-k}\,\Gamma(n+2)}{\Gamma(k+1)\Gamma(1+n-k)}\,dx$$

$$= \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(1+n-k)} \int_0^z x^k (1-x)^{n-k}\,dx$$

$$= \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(1+n-k)} \left(\frac{x^{k+1}\,{}_2F_1(k+1, k-n; k+2; x)}{k+1} \right)$$

$$\text{median} = x\Big|_{\text{CDF}=0.5}$$

$$\text{cred. low} = x\Big|_{\text{CDF}=0.025}$$

$$\text{cred. high} = x\Big|_{\text{CDF}=0.975}$$

${}_2F_1(k+1, k-n; k+2; x) =$ can calculate coeffs of first N terms once, then vary x, since we will solve for x with k, n = constant

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$$\mathrm{CDF} = \frac{\Gamma(n+2)\ x^{k+1}}{\Gamma(k+1)\,\Gamma(1+n-k)\,(k+1)}\ {}_2F_1(k+1, k-n, k+2; x)$$

Euler's transformation

$$\boxed{{}_2F_1(k+1, k-n, k+2; x)} = (1-x)^{n-k+1}\,{}_2F_1(1, n+2, k+2; x)$$

$\leftarrow$ A2

$${}_2F_1(1, n+2, k+2; x) = \frac{\Gamma(k+2)}{\Gamma(n+2)} \sum_{T=0}^{\infty} \frac{\Gamma(n+2+T)\, x^T}{\Gamma(k+2+T)}$$

$${}_2F_1(k+1, k-n, k+2; x) = (1-x)^{n-k+1}\, \frac{\Gamma(k+2)}{\Gamma(n+2)} \sum_{T=0}^{\infty} \frac{\Gamma(n+2+T)\, x^T}{\Gamma(k+2+T)}$$

$$\mathrm{CDF} = \left[\frac{\cancel{\Gamma(n+2)}\, x^{k+1}}{\Gamma(k+1)\Gamma(1+n-k)\,(k+1)}\right]\left[(1-x)^{n-k+1}\, \frac{\Gamma(k+2)}{\cancel{\Gamma(n+2)}} \sum_{T=0}^{\infty} \frac{\Gamma(n+2+T)\, x^T}{\Gamma(k+2+T)}\right]$$

simplify!

$$\mathrm{CDF} = \frac{x^{k+1}(1-x)^{n-k+1}\ \Gamma(k+2)}{\Gamma(k+1)\Gamma(1+n-k)\,(k+1)} \sum_{T=0}^{\infty} \frac{\Gamma(n+2+T)\, x^T}{\Gamma(k+2+T)}$$

This appears to diverge?

$$\mathrm{CDF} = \frac{x^{k+1}(1-x)^{n-k+1}\ \cancel{(k+1)}\,(k+1)!}{\Gamma(1+n-k)\,(k+1)\cancel{\ }\ \ k!} \sum_{T=0}^{\infty} \frac{\Gamma(n+2+T)\, x^T}{\Gamma(k+2+T)}$$

$\Gamma(n) = (n-1)! \quad \Gamma(n+1) = n!$ A 2

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

let $(p)_q = \dfrac{\Gamma(p+q)}{\Gamma(p)} \rightarrow$ rising factorial identity

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)\, z^n}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(n+1)}$$

~~let $a=1$, $b=n+2$, $c=k+2$,~~

let $n = T$

$${}_2F_1(a,b,c;z) = \sum_{T=0}^{\infty} \frac{\Gamma(a+T)\Gamma(b+T)\Gamma(c)\, z^T}{\Gamma(a)\Gamma(b)\Gamma(c+T)\Gamma(T+1)}$$

let $a=1$, $b=n+2$, $c=k+2$, $z=x$

$${}_2F_1(1,n+2,k+2;x) = \sum_{T=0}^{\infty} \frac{\cancel{\Gamma(T+1)}\Gamma(n+2+T)\Gamma(k+2)\, x^T}{\cancel{\Gamma(1)}\,\Gamma(n+2)\Gamma(k+2+T)\cancel{\Gamma(T+1)}}$$

$${}_2F_1(1,n+2,k+2;x) = \frac{\Gamma(k+2)}{\Gamma(n+2)} \sum_{T=0}^{\infty} \frac{\Gamma(n+2+T)\, x^T}{\Gamma(k+2+T)}$$

A3

$$CDF = \frac{x^{k+1}(1-x)^{n-k+1}\cancel{(k+1)}}{\Gamma(1+n-k)\cancel{(k+1)}} \sum_{T=0}^{\infty} \frac{\Gamma(n+2+T)\, x^{T}}{\Gamma(k+2+T)}$$ A3-1

$$CDF = \frac{x^{k+1}(1-x)^{n-k+1}}{\Gamma(1+n-k)} \sum_{T=0}^{\infty} \frac{\Gamma(n+2+T)\, x^{T}}{\Gamma(k+2+T)}$$ A3-2

$$CDF = \frac{x^{k+1}(1-x)^{n-k+1}}{\Gamma(1+n-k)} \sum_{T=0}^{\infty} \frac{(n+1+T)!\, x^{T}}{(k+1+T)!}$$

$$CDF = x^{k+1}(1-x)^{n-k+1} \sum_{T=0}^{\infty} \frac{(n+1+T)!\, x^{T}}{(n-k)!\,(k+1+T)!}$$

let $f(n,k,T) = \dfrac{(n+1+T)!}{(k+1+T)!}$

$$CDF = \frac{x^{k+1}(1-x)^{n-k+1}}{\Gamma(1+n-k)} \sum_{T=0}^{\infty} x^{T} f(n,k,T)$$

$$f(n,k,T) = \frac{(n+1+T)!}{(k+1+T)!} = \frac{\prod_{i=0}^{n+T}(n+T+1-i)}{\prod_{i=0}^{k+T}(k+T+1-i)} =$$

next page

$$\cancel{\mathcal{F}(n,k,T) = \prod_{i=0}^{k+T} (n+T+1-i) \prod_{i=}^{n-k}}$$

$$\mathcal{F}(n,k,T) = \frac{\prod_{i=1}^{n+T+1} i}{\prod_{i=1}^{k+T+1} i} = \frac{\prod_{i=1}^{(n-k)+k+T+1} i}{\prod_{i=1}^{k+T+1} i} \qquad \text{A4-1}$$

is this correct?
NO!

$$\mathcal{F}(n,k,T) = \frac{\cancel{\prod_{i=1}^{k+T+1} i} \; \prod_{\cancel{i=n-k}}^{n+T+1} i}{\cancel{\prod_{i=1}^{k+T+1} i}} \longrightarrow i=k+T+2 \qquad \text{A4-2}$$

$$\mathcal{F}(n,k,T) = \prod_{i=k+T+2}^{n+T+1} i \qquad \text{A4-3}$$

$$\text{CDF} = \frac{x^{k+1}(1-x)^{n-k+1}}{\Gamma(1+n-k)} \sum_{T=0}^{\infty} \left(x^{T} \prod_{i=k+T+2}^{n+T+1} i \right) \qquad \text{A4-4}$$